A Lower Confidence Limit for Reliability of a Coherent System with Independent Exponential Components

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Abstract:

In this paper, we solve an oldest open problem of finding interval estimate for system reliability via the CHA algorithm following the Easterling (Jour. Amer. Statist. Asso., 1972) approach . We consider a coherent system composed of exponential components; the components are assumed to be statistically independent. A closed form expression for the standard error of the system reliability, for a given mission of duration, is obtained. The method of calculating the 100(1-α)% lower confidence limit for the system reliability is illustrated for a simple series system with two independent exponential components. Both methods of simulation and numerical integration techniques are used for calculations. This paper basically extends the results of Easterling to any coherent system.

1. Introduction:

In this paper, we consider a coherent structure composed of independent exponential components. The problem is to find a lower confidence limit for the system reliability.We confine our attention to the lower confidence limit of system reliability, since it is of most interest to the reliability practitioners in the context of interval estimation of system reliability.

An excellent account on the topic of interval estimation of system reliability is available in Crowder et al. (1991). The first step in obtaining the lower confidence limit of a coherent system is to get a point estimate of the system reliability R(t). The basis of estimation of system reliability is the following model (under the assumption of independence of components, see Chaudhuri et al. (2001):

$$R(t) = \sum_{j=1}^{2^m-1} 1(j).\prod_{i=1}^{n} r_i(t)^{D(i,j)}, \qquad (1.1)$$

which connects the system reliability R(t) with the component reliabilities $r_i(t)$, $i = 1,\dots,n,$ for a mission of duration t. The notational details would be described in Section 2. Suppose we perform n_i binomial (success/failure) tests on component i and observe x_i successes, then $\frac{x_i}{n_i}$ is the maximum likelihood estimate of component reliability.

If $\hat{r}_i(t)$ denotes the MLE of ith component reliability, then the MLE of R(t) is given by:

$$\hat{R}(t) = \sum_{j=1}^{2^m-1} 1(j) \prod_{i=1}^{n} \hat{r}_i(t)^{D(i,j)} \qquad (1.2)$$

In principle, one can obtain $Var\left(\hat{R}\right)$ from (1.2), and thus, the lower

100(1-α)% confidence limit is calculated as

$$\hat{R}(t) - z_\alpha \sqrt{Var\left(\hat{R}\right)} \qquad (1.3)$$

where z_α is the α-fractile of the standard normal distribution (see Madansky, 1965).

Unfortunately, this does not work well, because the distribution of $\hat{R}$ is basically skewed and a skewed distribution is approximated by a symmetric distribution that is normal here. As a result the confidence intervals may be outside the interval [0,1] as noted by Easterling(1972). Dissatisfaction with this approximation led Easterling (1972) to consider the use of a binomial distribution with parameters

$\hat{n}$ and $\hat{R}$ where $\hat{n} = \dfrac{\hat{R}(1-\hat{R})}{Var(\hat{R})}$. This is justified in the light of the following observation: since $\hat{R}$ is a probability and it is obtained via binomial sampling, it is logical to treat $\hat{R}$ as a binomial estimate based on $\hat{n}$ trials.

Thus, the component test results can be thought of as being equivalent to system results of $\hat{n}$ tests with $\hat{n}\,\hat{R}$ successes. From this consideration one can now easily obtain a lower confidence limit for R(t). The required lower confidence limit is obtained from:

$$R_L = B\left(1-\alpha; x, \hat{n} - x + 1\right) \qquad (1.4)$$

where $B(\gamma; a, b)$ is the γ -fractile of the beta distribution with parameters a and b not being necessarily integers.

In this paper we address the problem of finding the lower confidence interval for the reliability of a complex system in the light of the CHA algorithm presented in Chaudhuri et al.(2001).

2. Exponential Components:

Suppose we have n_0 complete observations on each component. Let Z_i denote the total time on test at the last failure of the ith component. If $r_i(t) = \exp(-\lambda_i t)$ denotes the reliability of the ith component, then it is known that $2\lambda_i Z_i$ follows a χ^2 with $2n_0$ degrees of freedom. Now, an estimate of the system reliability is obtained as

$$\hat{R}(t) = \sum_{j=1}^{2^m-1} 1(j)\, e^{-\left\{\sum_{i=1}^{n} \hat{\lambda}_i D(i,j)\right\} t} \tag{2.1}$$

where $$\hat{\lambda}_i = \frac{n_0}{Z_i}. \tag{2.2}$$

Thus, letting $W_i \equiv 2\lambda_i Z_i$, we have

$$\hat{R}(t) = \sum_{j=1}^{2^m-1} 1(j)\, e^{-2n_0 t\left\{\sum_{i=1}^{n} \frac{\lambda_i D(i,j)}{W_i}\right\}} \tag{2.3}$$

Hence, $$E\,\hat{R}(t) = \sum_{j=1}^{2^m-1} 1(j) \prod_{i=1}^{n} E\left(e^{-\frac{2n_0 t \lambda_i D(i,j)}{W_i}} \right) \tag{2.4}$$

Similarly, one can obtain,

$$E\,\hat{R}^2 = \sum_{j=1}^{2^m-1} \prod_{i=1}^{n} E\left(e^{-\frac{4n_0 t \lambda_i D(i,j)}{W_i}} \right)$$

$$+ 2 \sum_{\substack{j<k \\ j,k=1}}^{2^m-1} 1(j)1(k) \prod_{i=1}^{n} E\left(e^{-\frac{2n_0 t \lambda_i (D(i,j)+D(i,k))}{W_i}} \right)$$

(2.5)

Since $$Var\left(\hat{R}\right) = E\,\hat{R}^2 - \left(E\,\hat{R}\right)^2, \tag{2.6}$$

$Var\left(\hat{R}\right)$ can be calculated from (2.4) and (2.5).

The computations involved in (2.6) are straightforward in the light of the CHA algorithm described in Chaudhuri et al (2001).

4. Illustrative Example:

Consider a series system composed of two independent exponential components with reliability function

$$e^{-\lambda_i t},\ i = 1,2. \tag{4.1}$$

The system has only one path set. Thus, see Chaudhuri et al (2001),

$$D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad 1 = (1) \tag{4.2}$$

For sake of illustration, we take $\lambda_1 = 1$ and $\lambda_2 = 2$.

In order to obtain $Var\left(\hat{R}\right)$ we apply two methods. Method of simulation and method of numerical integration.

The CHA algorithm is simple and easy-to-use. As expected, our approach here does fairly well in this simple study of series system with two
exponential components. This example is considered for illustrative purpose only. The CHA algorithm can handle any coherent structure of any complexity. Thus, we claim that we have solved one oldest estimation problem in system reliability. The advantages of the Easterling approach(1972) over the Likelihood Ratio approach (Madansky, 1965) was established by Easterling in 1972 paper where he demonstrated that his approach appears to be as accurate as the LR method and is considerably easier to implement to obtain the confidence limits. The cases (1) no distributional assumption is made about component lifetimes (2) component lifetimes are estimated by Kaplan Meier estimates are under study.

Table 1: Simulation and numerical results for the calculations of the lower confidence limit $R_L = \hat{B}(0.05; x, n - x + 1)$

	t	$\hat{R}$	**Var(** $\hat{R}$ **)**	x	$\hat{n - x + 1}$	R_L	$\hat{\lambda}_1$	$\hat{\lambda}_2$	**Mean Life**
Simu-lation	**.05**	**.8561**	**8.09E(-05)**	**1304.4058**	**220.2557**	**.8404**	**1.0358**	**2.0716**	**.33**
	.10	**.7512**	**.00043**	**328.2554**	**109.7369**	**.7148**	**.9537**	**1.9075**	
	.20	**.5194**	**.0017**	**75.7258**	**71.0531**	**.4481**	**1.0916**	**2.1832**	
	.30	**.3510**	**.0015**	**51.6392**	**96.4769**	**.2856**	**1.1632**	**2.3265**	
	.40	**.3111**	**.00045**	**145.5929**	**323.3475**	**.2758**	**.9729**	**1.9459**	
	.50	**.2546**	**.00053**	**90.6442**	**266.3252**	**.2169**	**.9119**	**1.8238**	
Numer-ical Integr-ation	**.05**	**.8607**	**.0010**	**99.0510**	**17.0298**	**.7962**	**1**	**2**	
	.10	**.7408**	**.0002**	**726.8368**	**255.2903**	**.7168**	**1**	**2**	
	.20	**.5488**	**.0002**	**690.6609**	**568.8053**	**.5252**	**1**	**2**	
	.30	**.4066**	**.0002**	**498.4469**	**728.5346**	**.3833**	**1**	**2**	
	.40	**.3012**	**3.53E(-07)**	**179461.8**	**416373.3**	**.3002**	**1**	**2**	
	.50	**.2231**	**1.4E(-07)**	**276296.54**	**961979.66**	**.2225**	**1**	**2**	

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